

ON THE SPEED OF A PLANAR RANDOM WALK AVOIDING ITS PAST CONVEX HULL

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ABSTRACT. We consider a random walk in \mathbb{R}^2 which takes steps uniformly distributed on the unit circle centered around the walker's current position but avoids the convex hull of its past positions. This model has been introduced by Angel, Benjamini and Virág. We show a large deviation estimate for the distance of the walker from the origin, which implies that the walker has positive \liminf speed.

1. INTRODUCTION

Angel, Benjamini and Virág introduced in [1] the following model of a random walk $(X_n)_{n \geq 0}$ in \mathbb{R}^2 , which they called the *rancher*. The walker starts at the origin $X_0 = 0$. Suppose it has already taken n steps ($n \geq 0$) and is currently at X_n . Then its next position X_{n+1} is uniformly distributed on the unit circle centered around X_n but conditioned so that the straight line segment $\overline{X_n, X_{n+1}}$ from X_n to X_{n+1} does not intersect the interior K_n^o of the convex hull K_n of the past positions $\{X_0, X_1, \dots, X_n\}$, see Figure 1. Note that $(X_n)_{n \geq 0}$ is not Markovian since in general one needs to know the whole history of the process in order to determine the transition probabilities for the next step. This makes this model difficult to analyse, a property it shares with many other self-interacting processes, see [1] and also [2] for references.

To the best of our knowledge, the only major rigorous result which has been proved so far for this model, see [1, Theorem 1], is that the walk has positive \limsup speed, i.e. there is a constant $c > 0$ such that P -a.s. $\limsup \|X_n\|/n > c$ as $n \rightarrow \infty$. Here (Ω, \mathcal{F}, P) is the underlying probability space.

The purpose of the present paper is to improve this result by showing the following.

2000 Mathematics Subject Classification. 60K35. Secondary 60G50, 52A22.

Key words: convex hull, large deviations, law of large numbers, random walk, self avoiding, speed.

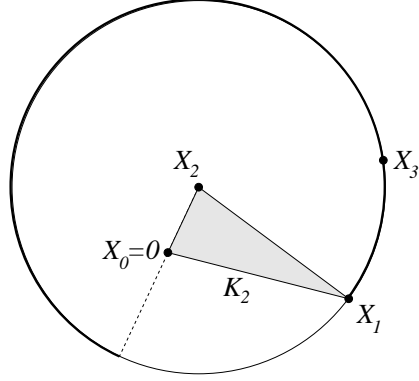


FIGURE 1. Three steps of the walk. X_3 is uniformly distributed on the bold arc of the circle with radius 1, centered in X_2 .

Theorem 1. *There is a constant $c_1 > 0$ such that*

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log P[\|X_n\| \leq c_1 n] < 0$$

and consequently,

$$(2) \quad \liminf_{n \rightarrow \infty} \frac{\|X_n\|}{n} \geq c_1 \quad P\text{-a.s.}$$

In particular, (2) proves [1, Conjecture 4]. We expect but were not able to prove that the speed $\lim \|X_n\|/n$ exists and is P -a.s. constant, as conjectured in [1, Conjecture 5]. For more conjectures regarding convergence of $X_n/\|X_n\|$ and transversal fluctuations of trajectories, see [1].

Let us now describe how the present article is organized. The next section introduces general notation. In Section 3 we introduce some sub- and supermartingales, which enable us in Section 4 to bound exponential moments of the time it takes the diameter of the convex hull K_n to increase. From this we deduce in Section 5 estimates for the diameter of K_n similar to the ones claimed in Theorem 1 for $\|X_n\|$ and show how this implies Theorem 1.

2. NOTATION

We denote by d_n the diameter of K_n . Since $(K_n)_n$ is an increasing sequence of sets, $(d_n)_n$ is non-decreasing. The ladder times τ_i at which the process $(d_n)_{n \geq 0}$ strictly increases are defined recursively by

$$\tau_0 := 0 \quad \text{and} \quad \tau_{i+1} := \inf\{n > \tau_i : d_n > d_{\tau_i}\} \quad (\leq \infty) \quad (i \geq 0).$$

Note that $\tau_1 = 1$ and that the τ_i 's are stopping times with respect to the canonical filtration $(\mathcal{F}_n)_{n \geq 0}$ generated by $(X_n)_{n \geq 0}$. Since the diameter of a bounded convex set is the distance between two of its extremal points there

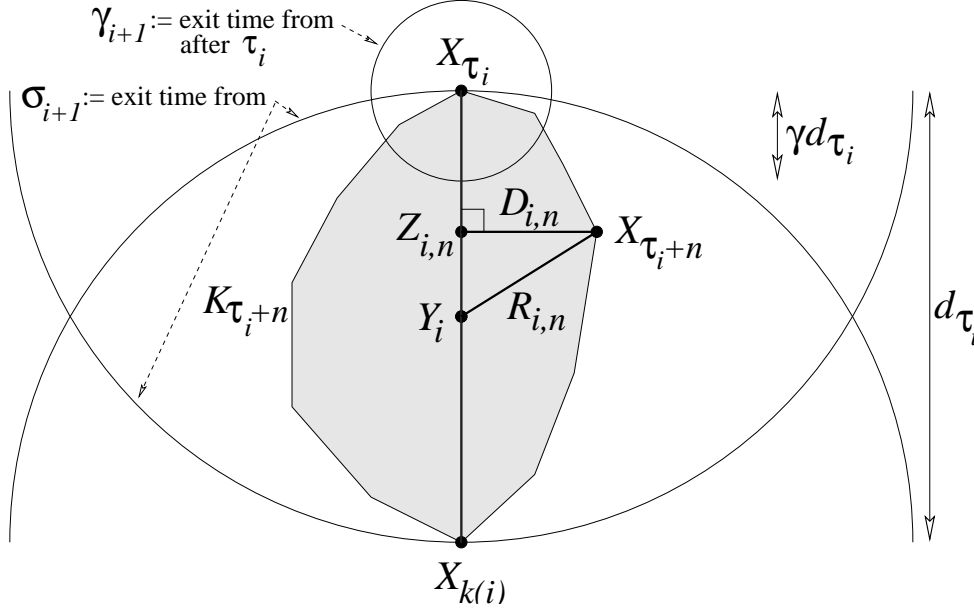


FIGURE 2. General notation

is for all $i \geq 1$ with $\tau_i < \infty$ a (P -a.s. unique) $0 \leq k(i) < \tau_i$ such that $d_{\tau_i} = \|X_{\tau_i} - X_{k(i)}\|$, see Figure 2. For $x \in \mathbb{R}^2$ and $r > 0$ we denote by $B(x, r)$ the closed disk with center x and radius r . If $\tau_i < \infty$ then

$$\sigma_{i+1} := \inf\{n \geq 0 \mid X_n \notin B(X_{\tau_i}, d_{\tau_i}) \cap B(X_{k(i)}, d_{\tau_i})\}$$

is the exit time of the walk from the large lens shaped region shown in Figure 2, which we shall refer to as the *lens created at time τ_i* . Observe that K_{τ_i} is contained in the lens created at time τ_i . Moreover,

$$(3) \quad \tau_{i+1} \leq \sigma_{i+1}$$

since if $\sigma_{i+1} < \infty$, $X_{\sigma_{i+1}}$ has a distance from either X_{τ_i} or $X_{k(i)}$ greater than d_{τ_i} . The point

$$(4) \quad Y_i := \frac{X_{\tau_i} + X_{k(i)}}{2} \quad (i \geq 1)$$

will serve as the “center” of K_{τ_i} and

$$R_{i,n} := \|X_{\tau_i+n} - Y_i\| \quad (i \geq 1, n \geq 0)$$

is the distance of X_{τ_i+n} from this center. The orthogonal projection of X_{τ_i+n} onto the straight line passing through X_{τ_i} and $X_{k(i)}$ will be called $Z_{i,n}$ ($i \geq 1, n \geq 0$). The distance of X_{τ_i+n} from this line is denoted by

$$D_{i,n} := \|X_{\tau_i+n} - Z_{i,n}\| \quad (i \geq 1, n \geq 0).$$

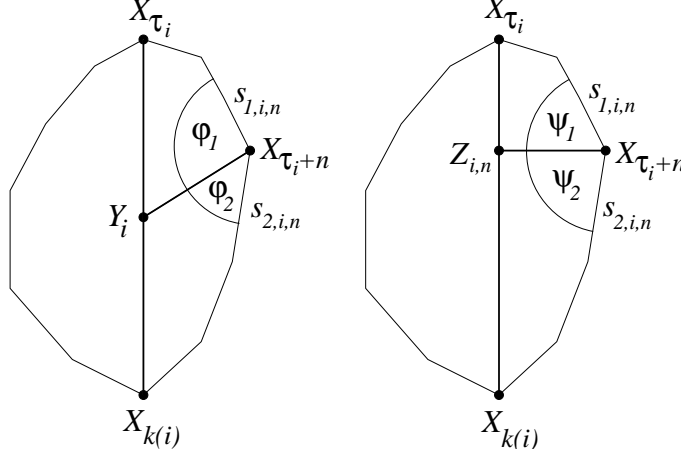


FIGURE 3. General notation

For the following definitions we assume $i, n \geq 1$ and $\tau_i + n < \tau_{i+1}$. In particular, due to (3), we assume that at time $\tau_i + n$ the walk has not yet left the lens created at time τ_i . This implies that $Z_{i,n} \in \overline{X_{\tau_i}, X_{k(i)}}$ and that X_{τ_i} and $X_{k(i)}$ are still boundary points of K_{τ_i+n} , as shown in Figures 2 and 3. Hence if we start in X_{τ_i+n} and follow the two boundary line segment emanating from X_{τ_i+n} we will eventually reach X_{τ_i} and $X_{k(i)}$. The boundary line segment whose continuation leads first to K_{τ_i} and then to $X_{k(i)}$ is called $s_{1,i,n}$, while the other line segment starting in X_{τ_i+n} is denoted by $s_{2,i,n}$, see Figure 3. The angle between $s_{j,i,n}$ and $\overline{X_{\tau_i+n}, Y_i}$ is called $\varphi_{j,i,n} \in [0, \pi]$ ($j = 1, 2$), see the left part of Figure 3. Similarly, the angle between $s_{j,i,n}$ and $\overline{X_{\tau_i+n}, Z_{i,n}}$ is denoted by $\psi_{j,i,n} \in [0, \pi]$ ($j = 1, 2$), see the right part of Figure 3. Occasionally, we will drop the subscripts i and n from φ and ψ . Since K_{τ_i+n} is convex,

$$(5) \quad \varphi_1 + \varphi_2 = \psi_1 + \psi_2 \leq \pi.$$

Furthermore, $|\varphi_1 - \psi_1|$ is one of the angles in a right angled triangle, namely the triangle with vertices X_{τ_i+n} , Y_i and $Z_{i,n}$. Hence,

$$(6) \quad |\varphi_1 - \psi_1| = |\varphi_2 - \psi_2| \leq \pi/2.$$

3. SOME SUB- AND SUPERMARTINGALES

The following result shows that for every $i \geq 1$, both $(R_{i,n})_n$ and $(D_{i,n})_n$ ($1 \leq n < \tau_{i+1} - \tau_i$) are submartingales.

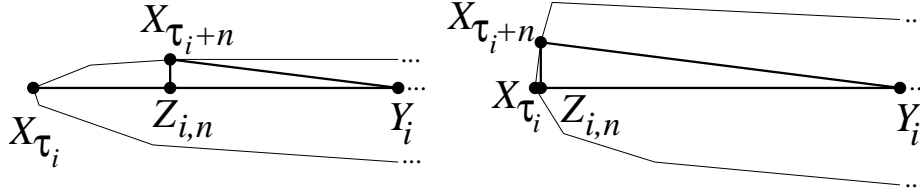


FIGURE 4. The expected increment of $R_{i,n}$ is small in the left figure and large in the right figure. For $D_{i,n}$ it is the other way round.

Lemma 2. For all $i, n \geq 1$, P -a.s. on $\{\tau_i + n < \tau_{i+1}\}$,

$$(7) \quad E[R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{\sin \varphi_{1,i,n} + \sin \varphi_{2,i,n}}{2\pi} \geq \frac{\sin \varphi_{1,i,n}}{2\pi} \geq 0,$$

$$(8) \quad E[D_{i,n+1} - D_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{\sin \psi_{1,i,n} + \sin \psi_{2,i,n}}{2\pi} \geq \frac{\sin \psi_{1,i,n}}{2\pi} \geq 0$$

and

$$(9) \quad E[D_{i,n+1} - D_{i,n} + R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq c_2$$

for some constant $c_2 > 0$.

Figure 4 shows examples in which the expected increments of $R_{i,n}$ and $D_{i,n}$ are close to 0, thus explaining, why we are not able to bound in (7) and (8) these expected increments individually away from 0. Note however, that in both situation depicted in Figure 4, if the expected increment of $R_{i,n}$ or of $D_{i,n}$ is small then the expected increment of the other quantity is large. This confirms that the expected increments of $R_{i,n}$ and $D_{i,n}$ cannot both be small at the same time, see (9).

Proof of Lemma 2. We fix $i, n \geq 1$ and drop them as subscripts of $\varphi_{j,i,n}$ and $\psi_{j,i,n}$ ($j = 1, 2$). Then the following statements hold on the event $\{\tau_i + n < \tau_{i+1}\}$. Consider the angle between $\overline{Y_i, X_{\tau_i+n}}$ and $\overline{X_{\tau_i+n}, X_{\tau_i+n+1}}$ which includes $s_{2,i,n}$. This angle is chosen uniformly at random from the interval $[\varphi_2, 2\pi - \varphi_1]$. Hence we get by a change of basis argument

$$\begin{aligned} & E[R_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] \\ &= \frac{1}{(2\pi - \varphi_1) - \varphi_2} \int_{\varphi_2}^{2\pi - \varphi_1} \|(R_{i,n}, 0) - (\cos \varphi, \sin \varphi)\| \, d\varphi \\ &\geq \frac{1}{2\pi - \varphi_1 - \varphi_2} \int_{\varphi_2}^{2\pi - \varphi_1} |R_{i,n} - \cos \varphi| \, d\varphi \\ &\geq R_{i,n} + \frac{1}{2\pi - \varphi_1 - \varphi_2} \int_{\varphi_2}^{2\pi - \varphi_1} -\cos \varphi \, d\varphi \\ &= R_{i,n} + \frac{\sin \varphi_1 + \sin \varphi_2}{2\pi - \varphi_1 - \varphi_2} \geq R_{i,n} + \frac{\sin \varphi_1 + \sin \varphi_2}{2\pi}, \end{aligned}$$

which shows (7). Similarly, (8) follows from

$$E[D_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] = \frac{1}{(2\pi - \psi_1) - \psi_2} \int_{\psi_2}^{2\pi - \psi_1} |D_{i,n} - \cos \psi| d\psi.$$

For the proof of (9) we assume without loss of generality $\varphi_1 \leq \pi/2$. Indeed, otherwise $\varphi_2 \leq \pi/2$ because of $\varphi_1 + \varphi_2 \leq \pi$, see (5), and in the following proof one only has to replace the subscript 1 by the subscript 2 and swap X_{τ_i} and $X_{k(i)}$. By (7) and (8),

$$(10) \quad E[D_{i,n+1} - D_{i,n} + R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{\sin \varphi_1 + \sin \psi_1}{2\pi}.$$

We will show that the right side of (10) is always greater than $c_2 := (4\pi^2)^{-1}$. Assume that it is less than c_2 . Then

$$(11) \quad \sin \varphi_1, \sin \psi_1 \leq (2\pi)^{-1}$$

and hence $\varphi_1 \leq (\pi/2) \sin \varphi_1 \leq 1/4$ by concavity of \sin on $[0, \pi/2]$. Similarly, (11) implies that either $\psi_1 \leq 1/4$ or $\pi - \psi_1 \leq 1/4$. Due to $|\varphi_1 - \psi_1| \leq \pi/2$, see (6), the latter case is impossible. Therefore,

$$(12) \quad |\varphi_1 - \psi_1| \leq \max\{|\varphi_1|, |\psi_1|\} \leq 1/4.$$

The angle $\alpha \in [0, \pi/2]$ between $\overline{Z_{i,n}, X_{\tau_i+n}}$ and $\overline{X_{\tau_i+n}, X_{\tau_i}}$ is less than or equal to ψ_1 . Consequently,

$$(13) \quad \sin \psi_1 \geq \sin \alpha = \frac{\|X_{\tau_i} - Z_{i,n}\|}{\|X_{\tau_i} - X_{\tau_i+n}\|} \geq \frac{\|X_{\tau_i} - Y_i\| - \|Y_i - Z_{i,n}\|}{d_{\tau_i}}.$$

However,

$$(14) \quad R_{i,n} = \|Y_i - X_{\tau_i+n}\| \stackrel{(4)}{\leq} (\|X_{\tau_i} - X_{\tau_i+n}\| + \|X_{k(i)} - X_{\tau_i+n}\|)/2 \leq d_{\tau_i}.$$

Therefore,

$$\sin \psi_1 \stackrel{(13)}{\geq} \frac{d_{\tau_i}/2}{d_{\tau_i}} - \frac{\|Y_i - Z_{i,n}\|}{\|Y_i - X_{\tau_i+n}\|} = \frac{1}{2} - \sin |\varphi_1 - \psi_1| \stackrel{(12)}{\geq} \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

which contradicts (11). \square

We fix the constants

$$(15) \quad \beta := 1 + 4\pi\sqrt{8} > 30 \quad \text{and} \quad \gamma := \frac{1}{2\beta} < \frac{1}{60}.$$

Whenever $\tau_i < \infty$ we denote the first exit time after τ_i from $B(X_{\tau_i}, \gamma d_{\tau_i})$ by

$$\gamma_{i+1} := \inf\{n > \tau_i : \|X_n - X_{\tau_i}\| > \gamma d_{\tau_i}\} \quad (\leq \infty),$$

see Figure 2. If $i \geq 1$ and $n \geq 0$ then we shall call n *good for i* if $n = 0$ or if

$$(16) \quad \tau_i + n < \tau_{i+1} \wedge \gamma_{i+1} \quad \text{and} \quad E[R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{1}{\pi\sqrt{8}} \text{ P-a.s..}$$

This means, $n \geq 1$ is good for i if at time $\tau_i + n$ the walker has not yet left the intersection of the small ball around X_{τ_i} and the lens shown in Figure 2 and, roughly speaking, feels a substantial centrifugal force pushing it away from the center Y_i . Good times help the walker to leave the lens shortly after τ_i and closely to the point X_{τ_i} . Next we introduce a family of supermartingales, which will help make this idea more precise.

Lemma 3. *There are constants $c_3 > 0, c_4 > 0$ and $1 \leq c_5 < \infty$ such that P -a.s. for all $i \geq 1, n \geq 0$,*

$$(17) \quad E[M_{i,n} \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_4 n),$$

where

$$(18) \quad M_{i,n} := \mathbf{1}\{\tau_i + n < \tau_{i+1}\} \times \exp\left(-c_3\left(D_{i,n} + \beta(R_{i,n} - R_{i,0}) - 4 \sum_{j=0}^{n-1} \mathbf{1}\{j \text{ is good for } i\}\right)\right).$$

Proof. Firstly, we shall prove that for suitable $c_4 > 0$,

$$(19) \quad E[M_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] \leq \exp(-c_4)M_{i,n} \quad P\text{-a.s. for all } i \geq 1, n \geq 1,$$

thus showing that $(M_{i,n})_{n \geq 1}$ is an exponentially fast decreasing submartingale for each $i \geq 1$. Fix $i \geq 1$ and $n \geq 1$. We have

$$\begin{aligned} E[M_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] &\leq M_{i,n} f_{i,n}(c_3) \quad P\text{-a.s., where} \\ f_{i,n}(c_3) &:= E[\exp(c_3 Z_{i,n}) \mid \mathcal{F}_{\tau_i+n}] \quad \text{and} \\ Z_{i,n} &:= D_{i,n} - D_{i,n+1} + \beta(R_{i,n} - R_{i,n+1}) \\ &\quad + 4\mathbf{1}\{n \text{ is good for } i\}. \end{aligned}$$

Therefore, in order to prove (19) we need to bound $f_{i,n}(c_3)$ on $\{\tau_i + n < \tau_{i+1}\}$ from above away from 1. By Taylor's expansion,

$$(20) \quad f_{i,n}(c_3) \leq 1 + c_3 E[Z_{i,n} \mid \mathcal{F}_{\tau_i+n}] + (c_3 c_6)^2 \exp(c_3 c_6)/2,$$

where $c_6 := 1 + \beta + 4$ is an upper bound for $Z_{i,n}$. On $\{\tau_i + n < \tau_{i+1}\}$, due to definition (15) of β ,

$$\begin{aligned} E[Z_{i,n} \mid \mathcal{F}_{\tau_i+n}] &= E[D_{i,n} - D_{i,n+1} + R_{i,n} - R_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] \\ &\quad + 4\left(\pi\sqrt{8}E[R_{i,n} - R_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] + \mathbf{1}\{n \text{ is good for } i\}\right) \\ &\leq -c_2 \end{aligned}$$

P -a.s. by virtue of Lemma 2 (7), (9), and definition (16). Consequently, we may and do choose $c_3 > 0$ small enough such that on $\{\tau_i + n < \tau_{i+1}\}$ the right hand side of (20) is P -a.s. less than a number strictly smaller than 1, which we call e^{-c_4} , thus showing (19). By induction over n we obtain

$$E[M_{i,n} \mid \mathcal{F}_{\tau_i}] \leq \exp(-c_4(n-1))E[M_{i,1} \mid \mathcal{F}_{\tau_i}] \quad P\text{-a.s.}$$

for all $n \geq 1$. Since $M_{i,1} \leq e^{c_3 c_6}$ this finishes the proof. One could do better by estimating $M_{i,1}$ more carefully, thus getting rid of the constant c_5 , but we do not need it. \square

4. EXPONENTIAL MOMENTS OF $\tau_{i+1} - \tau_i$.

We denote the differences between two successive finite ladder points of $(d_n)_n$ by $\Delta_i := \tau_{i+1} - \tau_i$ for $i \geq 0$.

Proposition 4. $\tau_i < \infty$ P -a.s. for all $i \geq 0$. Moreover, there are constants $1 \leq c_5 < \infty$ and $c_7 > 0$ such that for all $i \geq 0$ and $n \geq 0$, P -a.s.,

$$(21) \quad E[\exp(c_7 \Delta_i) \mid \mathcal{F}_{\tau_i}] \leq c_5 \quad \text{and}$$

$$(22) \quad P[\Delta_i \geq n \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_7 n).$$

Proof. We only need to show that there are constants $1 \leq c_5 < \infty$ and $c_7 > 0$ such that (21) holds for all $i \geq 1$ with $\tau_i < \infty$. Indeed, the case $i = 0$ is trivial since $\tau_0 = 0$, $\tau_1 = 1$ and hence $\Delta_0 = 1$. Moreover, (22) follows from (21) by Chebyshev's inequality and since (21) implies $\Delta_i < \infty$, we then have $\tau_i = \Delta_0 + \dots + \Delta_{i-1} < \infty$ as well.

Fix $i \geq 1$. For the proof of (21) we first show that with c_3, c_4 and c_5 according to Lemma 3 and $c_8 := c_3(1 + 2\beta)/c_4$ we have P -a.s. for all $i \geq 1, n \geq 0$,

$$(23) \quad P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(c_4(c_8 d_{\tau_i} - n)), \quad \text{and}$$

$$(24) \quad P[\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_4 n).$$

We shall show later how these auxiliary estimates imply (21). For $n = 0$, (23) and (24) are true since $c_5 \geq 1$. Fix $n \geq 1$. By (14) and the Pythagorean theorem, $D_{i,n} \leq R_{i,n} \leq d_{\tau_i}$ on the event $\{\tau_i + n < \tau_{i+1}\}$. Hence due to Lemma 3 (17) P -a.s. for all $n \geq 1$,

$$\exp(-c_3(1 + 2\beta)d_{\tau_i})P[\tau_i + n < \tau_{i+1} \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_4 n),$$

which is equivalent to (23). The second auxiliary statement (24) follows from Lemma 3 (17) once we have shown that on the event $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$,

$$(25) \quad D_{i,n} + \beta(R_{i,n} - R_{i,0}) - 4 \sum_{j=0}^{n-1} \mathbf{1}\{j \text{ is good for } i\} \leq 0.$$

First we will show that on $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$,

$$(26) \quad D_{i,n} + \beta(R_{i,n} - R_{i,0}) \leq 2(D_{i,n} - \|X_{\tau_i} - Z_{i,n}\|).$$

This is done by brute force. For abbreviation we set $d := d_{\tau_i}/2$, $y := D_{i,n}$ and $x := \|X_{\tau_i} - Z_{i,n}\|$ and note that on $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$ we have $x, y \in [0, \gamma d]$. Observe that x and y play the role of cartesian coordinates of

X_{τ_i+n} , see Figure 5. Using $R_{i,0} = d/2$ and $R_{i,n} = \sqrt{(d/2 - x)^2 + y^2}$ we see that (26) is equivalent to

$$\beta\sqrt{(d/2 - x)^2 + y^2} \leq y - 2x + \beta d/2.$$

Both sides of this inequality are nonnegative since x is less than γd , which is tiny compared to βd . Taking the square and rearranging shows that (26) is equivalent to

$$(27) \quad x(4x - 4y - 2\beta d - \beta^2 x + \beta^2 d) + y(y + \beta d - \beta^2 y) \geq 0.$$

Since $x, y \in [0, \gamma d]$ and $\beta\gamma = 1/2$, see (15), the terms $\beta^2 d$ in the first bracket and βd in the second bracket are the dominant terms, respectively, which shows that (27) and thus (26) holds. For the proof of (25) it therefore suffices to show that on $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$,

$$(28) \quad D_{i,n} - \|X_{\tau_i} - Z_{i,n}\| \leq 2 \sum_{j=0}^{n-1} \mathbf{1}\{j \text{ is good for } i\}.$$

Both $D_{i,j}$ and $-\|X_{\tau_i} - Z_{i,j}\|$ can increase by at most 1 if j increases by 1. Therefore, the left hand side of (28) is less than or equal to $2\#J$ where

$$J := \{0 \leq j < n \mid \forall 0 \leq m < j : D_{i,m} - \|X_{\tau_i} - Z_{i,m}\| \leq D_{i,j} - \|X_{\tau_i} - Z_{i,j}\|\}.$$

Hence it suffices to show that the elements of J are good for i . Note that $j = 0 \in J$ is good for i by definition of being good. So fix $1 \leq j \in J$. By Lemma 2 (7) it is enough to show that $\sin \varphi_{1,i,j} \geq 2^{-1/2}$, that is

$$(29) \quad \varphi_{1,i,j} \in [\pi/4, 3\pi/4].$$

On the one hand, $\varphi_1 - \psi_1$ is close to $\pi/2$, as can be seen in Figure 5. More precisely,

$$\begin{aligned} \sin(\varphi_1 - \psi_1) &= \frac{\|Y_i - Z_{i,j}\|}{\|Y_i - X_{\tau_i+j}\|} \geq \frac{\|Y_i - X_{\tau_i}\| - \|X_{\tau_i} - Z_{i,j}\|}{\|Y_i - X_{\tau_i}\| + \|X_{\tau_i} - X_{\tau_i+j}\|} \\ &\geq \frac{d_{\tau_i}/2 - \gamma d_{\tau_i}}{d_{\tau_i}/2 + \gamma d_{\tau_i}} = \frac{1 - 2\gamma}{1 + 2\gamma} \geq \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}. \end{aligned}$$

Since $0 \leq \psi_1 \leq \varphi_1$ this implies $\varphi_1 \geq \pi/4$. On the other hand, $\varphi_1 - \psi_1 \leq \pi/2$, see (6). Hence all that remains to be shown for the completion of the proof of (29) and (24) is that

$$(30) \quad \psi_{1,i,j} \leq \pi/4.$$

Consider the half line (dashed in Figure 5) starting at X_{τ_i+j} which includes an angle of $\pi/4$ with $\overline{X_{\tau_i+j}, Z_{i,j}}$ that contains $s_{1,i,j}$. We claim that this line does not intersect $K_{\tau_i+j}^o$. This would imply (30). To prove this claim observe that for any $c > 0$ the set of possible values for X_{τ_i+m} with $D_{i,m} - \|X_{\tau_i} - Z_{i,m}\| = c$ is a line parallel to the half line just described. Since $j \in J$ the walker did not

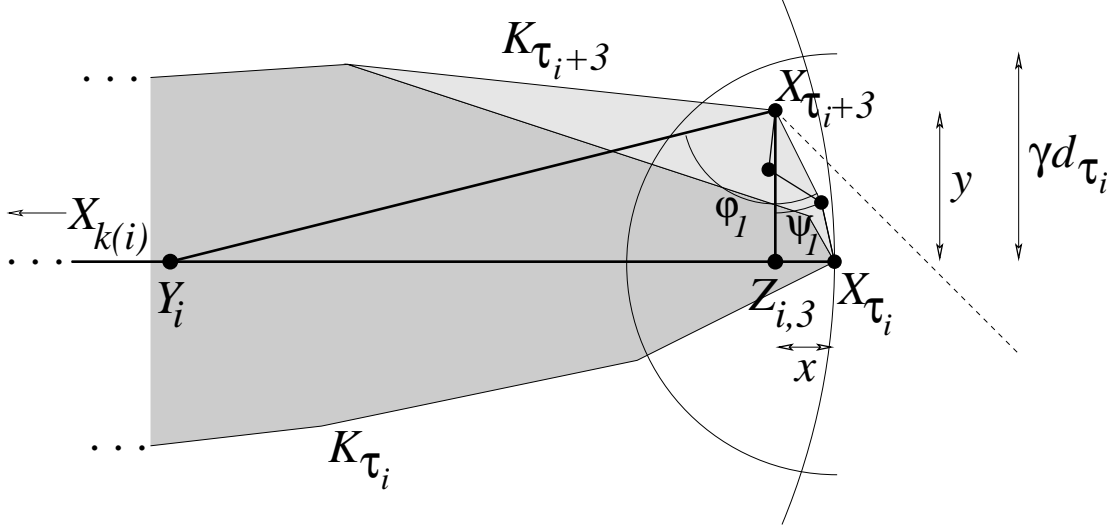


FIGURE 5. The darkly shaded convex hull K_{τ_i} at time τ_i has been enlarged after three steps by the lightly shaded part. $j = 3$ is good for i since it satisfies the sufficient criterion $\psi_{1,i,3} \leq \pi/4$, see (30), which corresponds to the fact that the dashed line, which intersects the horizontal axis at an angle of $\pi/4$, does not intersect $K_{\tau_i+3}^o$. $j = 1$ is also good for i for the same reason, while $j = 2$ might be good for i but fails to satisfy the sufficient condition (30), since the corresponding dashed line starting in X_{τ_i+2} would have intersected $K_{\tau_i+2}^o$.

cross between time τ_i and time τ_i+j-1 the dashed line passing through X_{τ_i+j} . Consequently, it suffices to show that the dashed line does not intersect $K_{\tau_i}^o$. If it did intersect $K_{\tau_i}^o$ then this would force the walker on its way from X_{τ_i} to X_{τ_i+j} to cross the dashed line strictly before time τ_i+j , which is impossible as we just saw. This completes the proof of (24).

Finally, we demonstrate how (23) and (24) imply (21) with

$$(31) \quad c_7 := \frac{\gamma c_4}{c_8 + \gamma}.$$

We distinguish three cases by partitioning Ω into three elements of \mathcal{F}_{τ_i} :

$$\{n \leq \gamma d_{\tau_i}\}, \quad \{\gamma d_{\tau_i} < n < (c_8 + \gamma)d_{\tau_i}\} \quad \text{and} \quad \{(c_8 + \gamma)d_{\tau_i} \leq n\}.$$

Note that

$$(32) \quad \gamma_{i+1} \geq \tau_i + \lceil \gamma d_{\tau_i} \rceil$$

since the walker takes steps of length one. Therefore, on $\{n \leq \gamma d_{\tau_i}\}$,

$$\begin{aligned} P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] &\stackrel{(32)}{=} P[\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \\ &\stackrel{(24)}{\leq} c_5 \exp(-c_4 n) \stackrel{(31)}{\leq} c_5 \exp(-c_7 n). \end{aligned}$$

On $\{\gamma d_{\tau_i} < n < (c_8 + \gamma)d_{\tau_i}\}$,

$$\begin{aligned} P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] &\leq P[\tau_i + \lceil \gamma d_{\tau_i} \rceil < \tau_{i+1} \mid \mathcal{F}_{\tau_i}] \stackrel{(32)}{=} P[\tau_i + \lceil \gamma d_{\tau_i} \rceil < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \\ &= \sum_{k \geq 1} P[\tau_i + k < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \mathbf{1}\{\lceil \gamma d_{\tau_i} \rceil = k\} \\ &\stackrel{(24)}{\leq} c_5 \exp(-c_4 \gamma d_{\tau_i}) \stackrel{(31)}{\leq} c_5 \exp(-c_7 n). \end{aligned}$$

Finally, on $\{(c_8 + \gamma)d_{\tau_i} \leq n\}$,

$$P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] \stackrel{(23)}{\leq} c_5 \exp(c_4(c_8 d_{\tau_i} - n)) \stackrel{(31)}{\leq} c_5 \exp(-c_7 n),$$

where the last inequality can easily be checked. \square

5. LINEAR GROWTH OF THE DIAMETER AND PROOF OF THEOREM 1

The following result (with $c = 0$) implies that $(d_n)_n$ has a positive lim inf speed.

Lemma 5. *There are constants $c_9 > 0$ and $c_{10} < \infty$ such that for all $n \geq 0$ and all $c \in [0, 1[$,*

$$(33) \quad E[\exp(d_{\lfloor cn \rfloor} - d_n)] \leq c_{10}(n+1) \exp(-c_9(1-c)n).$$

For the proof of this lemma and of Theorem 1 we need the following definition: Given $n \geq 0$ let $i_n := \sup\{i \geq 0 \mid \tau_i \leq n\}$. Note that

$$(34) \quad d_{\tau_{i_n}} = d_n \quad \text{and} \quad i_n \leq \tau_{i_n} \leq n < \tau_{i_n+1}.$$

Proof of Lemma 5. The case $n = 0$ is trivial. Now fix $n \geq 1$, $c \in [0, 1[$ and set $m = \lfloor cn \rfloor$,

$$(35) \quad c_{11} := \frac{1-c}{2} > 0 \quad \text{and} \quad c_{12} := \frac{c_7 c_{11}}{2 \ln c_5} > 0,$$

where c_5 and c_7 are according to Proposition 4. A simple union bound yields

$$\begin{aligned} E[e^{d_m - d_n}] &\leq \text{I} + \text{II} + \text{III}, \quad \text{where} \\ \text{I} &:= P[\tau_{i_m+1} - m \geq c_{11}n], \\ \text{II} &:= P[\tau_{i_m+1} - m < c_{11}n, i_n < i_m + \lceil c_{12}n \rceil] \quad \text{and} \\ \text{III} &:= E[\exp(d_m - d_n), i_n \geq i_m + \lceil c_{12}n \rceil], \end{aligned}$$

see also Figure 6. Here term I corresponds to the situation in which after

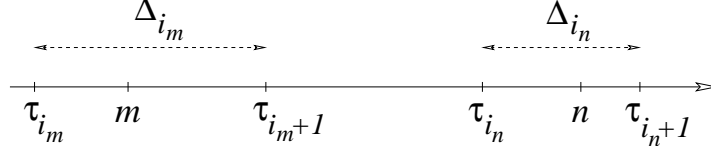


FIGURE 6.

time m the diameter does not increase for an untypical long while. Term II handles the case in which the diameter does increase shortly after time m , as it should, but not often enough in the remaining time until n . The third term III considers the original random variable on the typical event that the number of times at which the diameter increases is at least proportional to n with a constant of proportionality not too small.

It suffices to show that each of these three terms decays as $n \rightarrow \infty$ in the way stated in (33) with constants c_9 and c_{10} independent of c . As for the first term,

$$I \leq P[\Delta_{i_m} \geq \lceil c_{11}n \rceil] \stackrel{(34)}{\leq} \sum_{i=0}^m P[i_m = i, \Delta_i \geq \lceil c_{11}n \rceil] \stackrel{(22)}{\leq} c_5(n+1)e^{-c_7c_{11}n},$$

which is an upper bound like the one requested in (33). The second term is estimated as follows.

$$\begin{aligned}
 \text{II} &\stackrel{(34)}{=} P[\tau_{i_m+1} < \lfloor cn \rfloor + c_{11}n, n < \tau_{i_n+1} \leq \tau_{i_m+\lceil c_{12}n \rceil}] \\
 &\stackrel{(35)}{\leq} P[\tau_{i_m+\lceil c_{12}n \rceil} - \tau_{i_m+1} \geq (1 - c - c_{11})n = c_{11}n] \\
 &\leq E[\exp(c_7(\tau_{i_m+\lceil c_{12}n \rceil} - \tau_{i_m+1} - c_{11}n))] \\
 &= e^{-c_7c_{11}n} \sum_{k \geq 1} E[\exp(c_7(\tau_{k+\lceil c_{12}n \rceil-1} - \tau_k)), i_m + 1 = k] \\
 (36) \quad &= e^{-c_7c_{11}n} \sum_{k \geq 1} E\left[\prod_{i=0}^{\lceil c_{12}n \rceil-2} \exp(c_7\Delta_{k+i}), i_m + 1 = k\right].
 \end{aligned}$$

Note that $\{i_m + 1 = k\}$ is the event that τ_k is the first time after time m at which the diameter increases. Therefore,

$$(37) \quad \{i_m + 1 = k\} \in \mathcal{F}_{\tau_k}.$$

Moreover, the increments Δ_{k+i} are measurable with respect to $\mathcal{F}_{\tau_{k+i+1}}$. Consequently, by conditioning in (36) on $\mathcal{F}_{\tau_{k+\lceil c_{12}n \rceil-2}}$ and applying Proposition 4

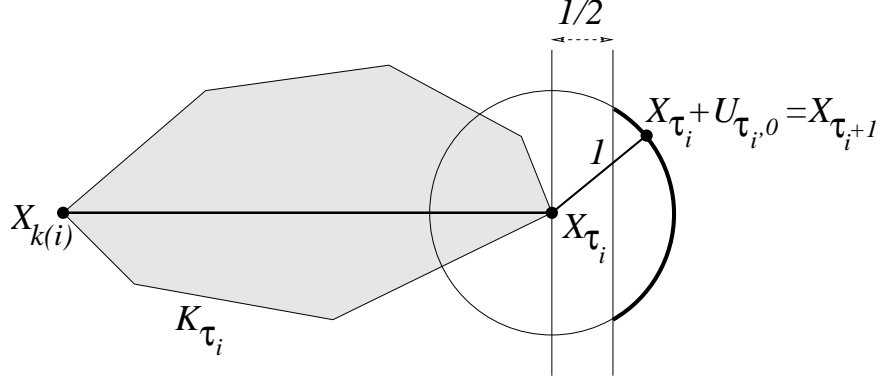


FIGURE 7. The event A_i occurs if the first trial point sampled lies on the bold arc.

(21) with $i = k + \lceil c_{12}n \rceil - 2$ we conclude

$$\text{II} \leq e^{-c_7 c_{11} n} c_5 \sum_{k \geq 1} E \left[\prod_{i=0}^{\lceil c_{12}n \rceil - 3} \exp(c_7 (\Delta_{k+i})), i_m + 1 = k \right].$$

Continuing in this way we obtain by induction after $\lceil c_{12}n \rceil - 1$ steps,

$$\text{II} \leq e^{-c_7 c_{11} n} c_5^{\lceil c_{12}n \rceil - 1} \leq e^{-c_7 c_{11} n} c_5^{c_{12}n} \stackrel{(35)}{=} e^{-(c_7/4)(1-c)n},$$

which is again of the form required in (33).

In order demonstrate that also the third term III behaves properly we will show that the increments $d_{\tau_{i+1}} - d_{\tau_i}, i \geq 1$, have a uniformly positive chance of being larger than a fixed constant, say $1/2$, independently of the past. More precisely, we may assume that the process $(X_n)_n$ is generated in the following way: There are i.i.d. random variables $U_{n,k}, n \geq 0, k \geq 0$, uniformly distributed on the unit circle centered in 0 such that $X_{n+1} = X_n + U_{n,k}$, where k is the smallest integer such that $\overline{X_n, X_n + U_{n,k}}$ does not intersect K_n^o . Then for any $i \geq 1$, by definition of τ_i ,

$$\begin{aligned} \{d_{\tau_{i+1}} \geq d_{\tau_i} + 1/2\} &\supseteq \{d_{\tau_{i+1}} \geq d_{\tau_i} + 1/2\} \\ (38) \quad &\supseteq \left\{ U_{\tau_i,0} \cdot \frac{X_{\tau_i} - X_{k(i)}}{d_{\tau_i}} \geq \frac{1}{2} \right\} =: A_i, \end{aligned}$$

see Figure 7. Here (38) holds for the following reason: Observe that for all $x \in K_{\tau_i}$,

$$\begin{aligned} (X_{\tau_i} - X_{k(i)}) \cdot \frac{X_{\tau_i} - X_{k(i)}}{\|X_{\tau_i} - X_{k(i)}\|} &= d_{\tau_i} \geq \|x - X_{k(i)}\| \\ &\geq (x - X_{k(i)}) \cdot \frac{X_{\tau_i} - X_{k(i)}}{\|X_{\tau_i} - X_{k(i)}\|} \end{aligned}$$

by Cauchy-Schwarz and thus

$$(39) \quad x \cdot (X_{\tau_i} - X_{k(i)}) \leq X_{\tau_i} \cdot (X_{\tau_i} - X_{k(i)}) \quad (x \in K_{\tau_i}).$$

However, on A_i ,

$$(X_{\tau_i} + U_{\tau_i,0}) \cdot (X_{\tau_i} - X_{k(i)}) \geq X_{\tau_i} \cdot (X_{\tau_i} - X_{k(i)}) + d_{\tau_i}/2 > X_{\tau_i} \cdot (X_{\tau_i} - X_{k(i)}).$$

Along with (39) this shows that $X_{\tau_i} + U_{\tau_i,0}$ and K_{τ_i} are lying on opposite sides of the line passing orthogonally to $\overline{X_{\tau_i}, X_{k(i)}}$ through X_{τ_i} . Therefore, $\overline{X_{\tau_i}, X_{\tau_i} + U_{\tau_i,0}}$ does not intersect $K_{\tau_i}^o$. Consequently, on A_i , $X_{\tau_{i+1}} = X_{\tau_i} + U_{\tau_i,0}$ and thus by Cauchy-Schwarz

$$d_{\tau_{i+1}} \geq \|X_{\tau_{i+1}} - X_{k(i)}\| \geq (X_{\tau_i} + U_{\tau_i,0} - X_{k(i)}) \cdot \frac{X_{\tau_i} - X_{k(i)}}{d_{\tau_i}} \geq d_{\tau_i} + \frac{1}{2},$$

which completes the proof of (38). Therefore, for all $1 \leq j_1 \leq j_2$,

$$(40) \quad d_{\tau_{j_2}} - d_{\tau_{j_1}} \geq \frac{1}{2} \sum_{i=j_1}^{j_2-1} \mathbf{1}\{d_{\tau_{i+1}} \geq d_{\tau_i} + 1/2\} \geq \frac{1}{2} \sum_{i=j_1}^{j_2-1} \mathbf{1}\{A_i\}.$$

This estimate will be useful since the random variables

$$(41) \quad \mathbf{1}\{A_i\} \ (i \geq 1) \quad \text{are i.i.d. with } P[A_i] > 0.$$

Indeed, let $\tilde{\mathcal{F}}_n$ ($n \geq 0$) be the σ -field generated by $U_{m,k}$, $0 \leq m < n$, $0 \leq k$. Because of $\mathcal{F}_n \subseteq \tilde{\mathcal{F}}_n$ we have $A_j \in \tilde{\mathcal{F}}_{\tau_i}$ for all $1 \leq j < i$. Moreover, since the uniform distribution on the unit circle is invariant under rotations,

$$(42) \quad A_i \text{ is independent of } \tilde{\mathcal{F}}_{\tau_i} \ (i \geq 1)$$

and $P[A_i | \tilde{\mathcal{F}}_{\tau_i}] = P[A_i]$ is just the length of the bold circle segment shown in Figure 7 divided by 2π . This implies (41). Now we estimate III by

$$(43) \quad \begin{aligned} \text{III} &\stackrel{(34)}{=} E \left[\exp(d_{\tau_{i_m}} - d_{\tau_{i_n}}), \ i_n \geq i_m + \lceil c_{12}n \rceil \right] \\ &\leq E \left[\exp(d_{\tau_{i_m+1}} - d_{\tau_{i_n}}), \ i_n \geq (i_m + 1) + \lceil c_{12}n \rceil - 1 \right] \\ &\leq \sum_{k \geq 1} E \left[\exp(d_{\tau_k} - d_{\tau_{k+\lceil c_{12}n \rceil-1}}), \ i_m + 1 = k \right] \\ &\stackrel{(40)}{\leq} \sum_{k \geq 1} E \left[\exp \left(-\frac{1}{2} \sum_{i=k}^{k+\lceil c_{12}n \rceil-2} \mathbf{1}\{A_i\} \right), i_m + 1 = k \right]. \end{aligned}$$

As seen in (37), $\{i_m + 1 = k\} \in \mathcal{F}_{\tau_k} \subseteq \tilde{\mathcal{F}}_{\tau_k}$. Therefore, after conditioning in (43) on \mathcal{F}_{τ_k} , we see with the help of (42) for $i \geq k$ and (41) that the right hand side of (43) equals

$$E \left[\exp \left(-\frac{1}{2} A_1 \right) \right]^{\lceil c_{12}n \rceil - 1},$$

which decays as required in (33), see (35). \square

Lemma 5 directly implies a weaker version of Theorem 1 in which $\|X_n\|$ is replaced by d_n . For the full statement we need the following additional argument.

Proof of Theorem 1. (2) follows from (1) by the Borel-Cantelli lemma. For the proof of (1) pick c_9 and c_{10} according to Lemma 5 and choose $c_{13} > 0$ and $c_1 > 0$ small enough such that

$$(44) \quad 2c_{13} - c_9 < 0 \quad \text{and} \quad 2c_1 - c_9(c_{13} - c_1) < 0.$$

We denote by $M_n := \max\{\|X_m\| \mid m \leq n\}$ the walker's maximal distance from the origin by time n . Note that M_n and d_n are related via

$$(45) \quad M_n \leq d_n \leq 2M_n \quad \text{for all } n \geq 0$$

because of $X_0 = 0$. By a union bound for any $n \geq 0$,

$$(46) \quad P[\|X_n\| \leq c_1 n] \leq P[\Delta_{i_n} \geq c_1 n] + P[M_n \leq c_{13} n] + P[B_n], \text{ where} \\ B_n := \{\Delta_{i_n} < c_1 n, M_n > c_{13} n, \|X_n\| \leq c_1 n\}.$$

It suffices to show that each one of the three terms on the right hand side of (46) decays exponentially fast in n . As for the first term,

$$P[\Delta_{i_n} \geq c_1 n] \stackrel{(34)}{\leq} \sum_{i=0}^n P[i_n = i, \Delta_i \geq c_1 n] \stackrel{(22)}{\leq} c_5(n+1)e^{-c_7 c_1 n},$$

which decays exponentially fast in n indeed. So does the second term in (46) since by Chebyshev's inequality,

$$P[M_n \leq c_{13} n] \stackrel{(45)}{\leq} P[d_n \leq 2c_{13} n] \leq e^{2c_{13} n} E[e^{-d_n}] \stackrel{(33)}{\leq} c_{10}(n+1)e^{(2c_{13}-c_9)n},$$

which decays exponentially fast due to the choice of c_{13} in (44). Finally, we are going to bound the third term in (46), $P[B_n]$. Define the ladder times $(\mu_j)_j$ of the process $(M_n)_{n \geq 0}$ recursively by

$$\mu_0 := 0 \quad \text{and} \quad \mu_{j+1} := \inf\{n > \mu_j \mid M_n > M_{\mu_j}\}.$$

In analogy to $(i_n)_n$ for $(\tau_i)_i$ we define for $(\mu_j)_j$ the increasing sequence $(j_n)_n$ by $j_n := \sup\{j \geq 0 \mid \mu_j \leq n\}$ and note that $\mu_{j_n} \leq n < \mu_{j_n+1}$ and $M_n = \|X_{\mu_{j_n}}\|$. Hence on the event B_n ,

$$\|X_{\mu_{j_n}} - X_n\| \geq \|X_{\mu_{j_n}}\| - \|X_n\| = M_n - \|X_n\| \geq (c_{13} - c_1)n.$$

Since the walker takes steps of length one, this implies $n - \mu_{j_n} \geq (c_{13} - c_1)n$ and therefore, on the event B_n ,

$$(47) \quad \mu_{j_n} \leq \lfloor (1 - c_{13} + c_1)n \rfloor.$$

On the other hand, on B_n ,

$$\begin{aligned}
d_n &\stackrel{(34)}{=} d_{\tau_{i_n}} = \|X_{\tau_{i_n}} - X_{k(i_n)}\| \leq \|X_n\| + \|X_n - X_{\tau_{i_n}}\| + \|X_{k(i_n)}\| \\
&\leq c_1 n + \Delta_{i_n} + M_n \leq c_1 n + c_1 n + M_{\mu_{j_n}} \\
&\stackrel{(45)}{\leq} 2c_1 n + d_{\mu_{j_n}} \stackrel{(47)}{\leq} 2c_1 n + d_{\lfloor (1-c_{13}+c_1)n \rfloor},
\end{aligned}$$

where we used in the second inequality again the fact that the steps have length one. Therefore, by Chebyshev's inequality and (33),

$$P[B_n] \leq P[d_n - d_{\lfloor (1-c_{13}+c_1)n \rfloor} \leq 2c_1 n] \leq c_{10}(n+1)e^{(2c_1-c_9(c_{13}-c_1))n},$$

which decays exponentially in n due to the choice of c_{13} and c_1 in (44). \square

REFERENCES

- [1] O. ANGEL, I. BENJAMINI and B. VIRÁG. Random walks that avoid their past convex hull. *Elect. Comm. in Probab.* **8** (2003) 6–16.
- [2] I. BENJAMINI and D.B. WILSON. Excited random walk. (2003) Preprint

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